

# Linear quadratic stochastic control problems with singular stochastic terminal constraint

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## Abstract

We provide a probabilistic solution to a linear quadratic optimal stochastic control problem with stochastic coefficients and a possibly singular stochastic terminal state constraint on a set with positive but not necessarily full probability. The analysis of such a control problem arises from optimal tracking problems of a given predictable target process where the terminal position is also constrained to match a specific exogenously prescribed random target level on a certain set of scenarios. The main novelty of our contribution is the characterization of the optimal control and the corresponding optimal value by an optimal signal process which reveals not only necessary and sufficient conditions under which the problem admits a finite value, but also allows us to tackle the delicate random singularity at terminal time via a suitable time consistent approximation of the optimization problem.

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## 1 Introduction

Linear quadratic stochastic optimal control problems (stochastic LQ problems in short) represent an important class of stochastic control problems

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and are very well studied in the literature, cf., e.g., the book by Yong and Zhou [24], Chapter 6, for an overview. A prototype of a stochastic LQ problem with linear quadratic cost functional is given by the so-called *optimal follower* or *optimal tracking* problem where one seeks to minimize a cost criterium of the following form: For a deterministic time horizon  $T > 0$ , for a predictable target process  $(\xi_t)_{0 \leq t \leq T}$  as well as progressively measurable, non-negative processes  $(\nu_t)_{0 \leq t \leq T}$  and  $(\kappa_t)_{0 \leq t \leq T}$ , for random variables  $\eta$  and  $\Xi_T$  known at time  $T$  and  $x \in \mathbb{R}$ , find a control  $u$  with state process

$$X_t^u = x + \int_0^t u_t dt \quad (0 \leq t \leq T) \quad (1)$$

which minimizes the objective

$$\mathbb{E} \left[ \int_0^T (X_t^u - \xi_t)^2 \nu_t dt + \int_0^T \kappa_t u_t^2 dt + \eta (X_T^u - \Xi_T)^2 \right]. \quad (2)$$

The interpretation of such a LQ problem is the following: The first term in (2) measures the overall quadratic deviation of the controlled state process  $X^u$  from the target process  $\xi$  weighted with a stochastic weight process  $\nu$ . The second term in (2) measures the incurred tracking effort in terms of running quadratic costs which are imposed on the control  $u$  with stochastic cost process  $\kappa$ . The third term in (2) implements a penalization on the quadratic deviation of the controlled state  $X_T^u$  from the final target position  $\Xi_T$  at terminal time  $T$  with nonnegative random penalization parameter  $\eta$ . It is well known in the literature that the optimal control to such a stochastic LQ problem as well as its optimal value is fully characterized by two coupled backward stochastic differential equations (BSDEs): A backward stochastic *Riccati* differential equation (BSRDE) of the form

$$dc_t = \left( \frac{c_t^2}{\kappa_t} - \nu_t \right) dt - dN_t \quad \text{on } [0, T] \text{ with } c_T = \eta \quad (3)$$

and, due to the linear component in the objective functional in (2), a linear BSDE of the form

$$db_t = \left( \frac{c_t}{\kappa_t} b_t - \nu_t \xi_t \right) dt + dM_t \quad \text{on } [0, T] \text{ with } b_T = \eta \Xi_T, \quad (4)$$

where  $(N_t)_{0 \leq t \leq T}$ ,  $(M_t)_{0 \leq t \leq T}$  denote some càdlàg local martingales (cf., e.g., Kohlmann and Tang [14], Section 5.1).

In the present paper, we are interested in the stochastic LQ problem in (2) which additionally incorporates a *possibly singular stochastic* terminal state constraint. Specifically, we allow the random penalization parameter  $\eta$  at terminal time to take the value infinity with positive (not necessarily full) probability. On the event  $\{\eta = +\infty\}$ , it is intuitively sensible to expect that the “blow up” of  $\eta$  imposes a *stochastic* terminal state constraint of the form

$$X_T^u = \Xi_T \quad \text{a.e. on the set } \{\eta = +\infty\} \quad (5)$$

on all controlled processes  $X^u$  that produce a finite value in (2). Mathematically, it is less obvious how to tackle this delicate *partial* singularity and how to compute the optimal control as well as the optimal value: First, note that the involved BS(R)DEs in (3) and (4) will both now exhibit with positive probability a singularity at final time in this case. The possibly singular BSRDE in (3) does not pose a problem. Its solvability has been recently studied in Kruse and Popier [15] and Popier [20], and we will assume the solution process to be given. In contrast, the singularity in the terminal condition of the linear BSDE in (4) is rather unpleasant because it also involves the desired target position  $\Xi_T$ , leading to an ill-posed problem in general. As a consequence, one needs to find a suitable substitute for the linear BSDE. Second, it is not clear a priori whether a controlled process  $X^u$  which respects the stochastic constraint in (5) and matches the random position  $\Xi_T$  on the event  $\{\eta = +\infty\}$  at terminal time  $T$  does likewise entail *finite* expected quadratic costs in (2). Put differently, the problem might not admit an optimal finite solution at all. This has to be precluded via identifying appropriate conditions.

We will answer the above questions and provide a probabilistic solution to the LQ problem in (2) with singular stochastic terminal state constraint in the sense of (5) by adopting the results recently obtained in Bank et al. [4] in the case of constant coefficients and almost sure terminal state constraint. The work in [4] revealed that the optimal solution and the corresponding optimal value can be characterized in a particularly enlightening manner by a specific *optimal signal process*: The optimally controlled state process reverts towards the latter and the signal makes transparent how the regularity and predictability of the targets  $\xi$  and  $\Xi_T$  determine the problem’s value. In the present more general setting, given the solvability of the singular BSRDE in (3), it turns out that the same key role is played by a generalized version of this optimal signal process. Indeed, loosely speaking, this process is given by the ratio  $b/c$  of the solution processes of the BS(R)DEs in (4) and (3).

Note that the ratio admits the desired terminal target position  $\Xi_T$  on the set  $\{\eta > 0\}$ . In particular, this optimal signal process will serve as an adequate substitute for the linear BSDE in (4). As we will see below it also allows similarly to [4] for an intuitively appealing interpretation of the optimal control and makes transparent the associated optimal costs.

This signal process, together with the solution process  $c$  of the singular BSRDE in (3), will provide the main tool not only in solving but also in tackling the considered LQ problem with its delicate stochastic terminal state constraint. Our main idea is to resolve the technical difficulties due to the singular state constraint by *moving away* in a suitable manner from terminal time  $T$  and to approximate the LQ problem in (2) via a consistent *truncation in time*. Specifically, we propose to consider the original problem as a limit of “tame” stochastic LQ problems with a strictly shorter time horizon  $\tau < T$  at which we impose a terminal penalization term with “classical” finite coefficient. The optimal signal process will be the proper key ingredient for choosing these penalizations in a *time consistent* manner. It turns out that the optimal controls and the corresponding costs to these time truncated LQ problems can be identified and prolonged without difficulties to the original terminal time  $T$  where the desired stochastic state constraint is satisfied. As a byproduct, we obtain under minimal assumptions (given the solvability of the singular BSRDE in (3)) necessary and sufficient conditions in terms of the optimal signal process and the solution process of the singular BSRDE in (3) under which the LQ problem with singular stochastic terminal state constraint actually admits a finite optimal value.

Stochastic control problems, referred to as optimal liquidation problems in the literature, with almost sure singular (i.e.,  $\eta = +\infty$  almost surely) and deterministic terminal state constraint (targeting the terminal position  $\Xi_T = 0$ ), where the cost functional is allowed to be quadratic in the state process  $X^u$  and the control  $u$  (that is,  $\xi \equiv 0$  in (2)) have already been studied in, e.g., Schied [23], Ankirchner et al. [3] and, in a more general BSPDE framework, in Graewe et al. [11]; allowing the penalization parameter  $\eta$  to take the value infinity with positive probability has been investigated in Kruse and Popier [15]. Ankirchner and Kruse [2], still within this context of optimal liquidation, allow the objective functional to be additionally linear in the control  $u$ . They also incorporate a specific non-zero stochastic terminal state constraint where the random target position  $\Xi_T$  is gradually revealed up to terminal time  $T$ . A general class of stochastic control problems including LQ problems with terminal states being constrained to a convex set were studied

by Ji and Zhou [12]. However, to the best of our knowledge, stochastic linear quadratic control problems with  $\xi \neq 0$  and possibly singular, general stochastic terminal state constraint  $\Xi_T \neq 0$  as considered in the present paper have not yet been investigated.

The analysis of the stochastic LQ problem in (2) above is especially motivated by optimal trading and hedging problems in Mathematical Finance. In this framework the state process  $X^u$  denotes an agent's position in some risky asset that she trades at a turnover rate  $u$ . She wants her position to be as close as possible to a given target strategy  $\xi$  but simultaneously seeks to minimize the induced quadratic transaction costs which are levied on her transactions due to, e.g., stochastic price impact as measured by  $\kappa$ . The weight process  $\nu$  captures stochastic volatility, that is, the risk of her open trading position due to random market fluctuations. Finally, in case of a possible but not necessarily almost sure occurrence of specific market conditions, encoded by the event set  $\{\eta = +\infty\}$ , she may require to drive her position  $X^u$  imperatively towards a predetermined random value  $\Xi_T$  at maturity  $T$  (e.g., to respect specific requirements of contractual or regulatory nature concerning her risky asset position). Otherwise, a penalization depending on the deviation of  $X_T^u$  from the target position  $\Xi_T$  is implemented. We refer to, e.g., Rogers and Singh [22], Naujokat and Westray [17], Almgren and Li [1], Frei and Westray [10], Cartea and Jaimungal [8], Cai et al. [7], Bank et al. [4] and, for asymptotic considerations, to Chan and Sircar [9]. Note, however, that the above cited papers may neither allow for an arbitrary predictable target strategy  $\xi$  nor for stochastic price impact  $\kappa$  and stochastic volatility  $\nu$ . In particular, none of them consider a possibly singular stochastic terminal state constraint in the sense of (5) above with general random target position  $\Xi_T$ .

The rest of the paper is organized as follows. In Section 2 we formulate the stochastic LQ problem with singular stochastic terminal state constraint. Our main result, the solution to the problem, is presented in Section 3. The proofs are deferred to Section 4.

## 2 Problem formulation

We fix a finite deterministic time horizon  $T > 0$  and a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \leq t \leq T}, \mathbb{P})$  satisfying the usual conditions of right continuity and completeness. We let  $(\kappa_t)_{0 \leq t \leq T}$  and  $(\nu_t)_{0 \leq t \leq T}$  denote two progressively

measurable, strictly positive processes such that

$$\int_0^T \left( \nu_t + \frac{1}{\kappa_t} \right) dt < \infty \quad \mathbb{P}\text{-a.s.} \quad (6)$$

In addition, we are given a predictable process  $(\xi_t)_{0 \leq t \leq T}$  satisfying

$$\mathbb{E} \left[ \int_0^T \xi_t \nu_t dt \right] < \infty \quad \text{and} \quad \int_0^T \xi_t^2 \nu_t dt < \infty \quad \mathbb{P}\text{-a.s.} \quad (7)$$

as well as a random terminal position  $\Xi_T \in L^0(\mathbb{P}, \mathcal{F}_{T-})$ . The random penalization parameter  $\eta \in \mathcal{F}_T$  satisfies

$$\mathbb{P}[0 \leq \eta \leq +\infty] = 1, \quad (8)$$

i.e.,  $\eta$  is supposed to be nonnegative  $\mathbb{P}$ -a.s. and possibly takes the value  $+\infty$  with positive probability.

*Remark 2.1.* The mild integrability conditions in (6) and (7) ensure that all the processes to be introduced shortly are well defined along with our stochastic LQ problem.

## 2.1 Connection between stochastic LQ problems and BS(R)DEs

It is well known that the solution to a stochastic LQ problem is intimately related to the solution of backward stochastic Riccati differential equation (BSRDE), cf., e.g., Bismut [5], [6]. In addition, a terminal state constraint in the LQ problem typically leads to a singular terminal condition for the corresponding BSRDE, cf., e.g., Ankirchner et al. [3], Kruse and Popier [15] or Graewe et al. [11]. The following standing assumption summarizes what we need to know about this BSRDE for our purposes:

**Assumption 2.2.** There exists a unique  $(\mathcal{F}_t)_{0 \leq t < T}$ -adapted, càdlàg semimartingale  $(c_t)_{0 \leq t < T}$  with BSRDE dynamics

$$dc_t = \left( \frac{c_t^2}{\kappa_t} - \nu_t \right) dt - dN_t \quad \text{on } [0, T) \quad (9)$$

for some càdlàg local martingale  $(N_t)_{0 \leq t < T}$  and (possibly) singular terminal condition

$$\lim_{t \uparrow T} c_t = \eta \quad \mathbb{P}\text{-a.s.} \quad (10)$$

The pair  $(c, N)$  satisfies

$$\mathbb{E} \left[ \sup_{s \in [0, t]} |c_s|^2 + [N]_t \right] < \infty \text{ for all } 0 \leq t < T. \quad (11)$$

Moreover, it holds that

$$c_t > 0 \quad \mathbb{P}\text{-a.s. for all } 0 \leq t < T \quad (12)$$

and

$$\int_{[0, T)} \frac{d[c]_t}{c_{t-}^2} < \infty \quad \text{on the set } \{\eta = +\infty\}, \quad (13)$$

where  $[c]$  denotes the quadratic variation process of the càdlàg semimartingale  $c$  (cf., e.g., Protter [21], Chapter II.6, for the quadratic variation process of càdlàg semimartingales).

*Remark 2.3.* 1. Note that the dynamics in (9) have to be understood in the sense that for all  $0 \leq s \leq t < T$  the pair  $(c, N)$  satisfies

$$c_s = c_t - \int_s^t \left( \frac{c_u^2}{\kappa_u} - \nu_u \right) du + \int_s^t dN_u \quad \mathbb{P}\text{-a.s.}$$

In particular, the dynamics in (9) are only required to hold on  $[0, T - \varepsilon]$  for every  $\varepsilon > 0$ , that is, strictly before  $T$ .

2. Let us mention that in the special *non-singular* case where the random variable  $\eta$  is simply bounded  $\mathbb{P}$ -a.s., existence and uniqueness results (within a Brownian framework and for bounded processes  $(\nu_t)_{0 \leq t \leq T}$  and  $(\kappa_t)_{0 \leq t \leq T}$ ) to the above BSRDE in (9) with terminal condition  $c_T = \eta$   $\mathbb{P}$ -a.s. can be found in Kohlmann and Tang [14]. The corresponding solution pair  $(c, N)$  satisfies property (11). Sufficient conditions under which the solution process  $(c_t)_{0 \leq t \leq T}$  is strictly positive, i.e., condition (12) holds true, are also provided therein.
3. In the singular case  $\eta = +\infty$   $\mathbb{P}$ -a.s. and again within a Brownian framework, existence and uniqueness results (under suitable integrability conditions on the processes  $(\nu_t)_{0 \leq t \leq T}$  and  $(\kappa_t)_{0 \leq t \leq T}$ ) to the above BSRDE in (9) with singular terminal condition  $\lim_{t \uparrow T} c_t = +\infty$   $\mathbb{P}$ -a.s. are provided in Ankirchner et al. [3] (cf. also Graewe et al. [11]). Therein, the solution pair  $(c, N)$  satisfies likewise conditions (11) and (12).

4. In the present partial singular setup where the random variable  $\eta$  is allowed to take the value  $+\infty$  with positive probability but not necessarily  $\mathbb{P}$ -a.s., Kruse and Popier [15] provide sufficient conditions (including suitable integrability conditions on  $(\kappa_t)_{0 \leq t \leq T}$  and  $(\nu_t)_{0 \leq t \leq T}$ ) for the existence of a minimal weak supersolution to the above BSRDE in (9) with slightly weaker (possibly) singular terminal condition  $\liminf_{t \uparrow T} c_t \geq \eta$   $\mathbb{P}$ -a.s. (cf. also the appendix below). Sufficient conditions on the random variable  $\eta$  and the process  $(\kappa_t)_{0 \leq t \leq T}$  under which the solution process  $(c_t)_{0 \leq t < T}$  possesses a left limit as  $t \uparrow T$  which is equal to  $\eta$  (as required in (10)) are discussed in Popier [20]. The solution pair  $(c, N)$  provided in [15] satisfies the integrability condition in (11) and the solution process  $(c_t)_{0 \leq t < T}$  is shown to be nonnegative. In the appendix below, we will provide within the setup of [15] a simple lower bound on the solution process which gives sufficient conditions under which strict positivity of the process  $c$  on  $[0, T)$ , i.e., property (12), is guaranteed (cf. Lemma 4.3 below).
5. Concerning the integrability condition in (13) on the “blow up” set  $\{\eta = +\infty\}$ , it is implicitly shown in Popier [19] in a Brownian framework and in the special case of constant coefficients  $\nu \equiv 0$  and  $\kappa \equiv 1$  that this condition is indeed satisfied by the solution process  $(c_t)_{0 \leq t < T}$  of the corresponding BSRDE (9) (cf. Theorem 2 and Proposition 3 in [19]). The other above cited papers [15], [20] and [3] do not further investigate this property, though. Since the required integrability in (13) is needed in the proof of Lemma 2.4 below whose result feeds crucially into our solution presented in Section 3, we will briefly discuss exemplarily in the appendix within the framework of [15] sufficient conditions on  $(\kappa_t)_{0 \leq t \leq T}$ ,  $(\nu_t)_{0 \leq t \leq T}$  and  $\eta$  under which property (13) does hold true.

Due to the linear component in the objective function of stochastic LQ problems, it is also well known in classical literature that the characterization of the optimal control and the optimal value requires additionally to the BSRDE in (9) a linear BSDE, cf., e.g., Kohlmann and Tang [14], Section 5. As discussed in the introduction above, it is suitable in our setup to introduce the so called *optimal signal process* instead. Inspired by the results in Bank et al. [4], it will be our main tool in tackling and solving our LQ problem with singular stochastic terminal state constraint and substitutes the classical linear BSDE. In order to introduce this process, let us first define



the *adjoint process*

$$L_t \triangleq c_t e^{-\int_0^t \frac{c_u}{\kappa_u} du} \quad (0 \leq t < T). \quad (14)$$

We refer to the discussion after Definition 2.5 below for an explanation of this terminology.

**Lemma 2.4.** *The adjoint process  $(L_t)_{0 \leq t < T}$  is a strictly positive càdlàg supermartingale. In particular,*

$$L_T \triangleq \lim_{t \uparrow T} L_t \geq 0 \quad \text{exists } \mathbb{P}\text{-a.s.} \quad (15)$$

and we have that  $(L_t)_{0 \leq t \leq T}$  is a supermartingale on  $[0, T]$ . Moreover, we have  $\{L_T = 0\} = \{\eta = 0\}$  up to  $\mathbb{P}$ -null sets.

*Proof.* Since  $c_t > 0$   $\mathbb{P}$ -a.s. for all  $0 \leq t < T$  by condition (12), it is immediate from (14) that also  $L_t > 0$   $\mathbb{P}$ -a.s. for all  $0 \leq t < T$ . Integration by parts and using the dynamics of  $c$  in (9) yields that  $L$  satisfies the dynamics

$$L_0 = c_0, \quad dL_t = L_{t-} \left( -\frac{\nu_t}{c_{t-}} dt - \frac{1}{c_{t-}} dN_t \right) \quad \text{on } [0, T]. \quad (16)$$

Since  $N$  is a càdlàg local martingale on  $[0, T]$ , we obtain from (16) that the process  $L$  is a càdlàg supermartingale on  $[0, T]$ . Hence, it follows by the (super-)martingale convergence theorem (see, e.g., Karatzas and Shreve [13], Chapter 1.3, Problem 3.16) that the limit  $L_T \triangleq \lim_{t \uparrow T} L_t$  exists  $\mathbb{P}$ -a.s. and extends the process  $L$  to a càdlàg supermartingale on all of  $[0, T]$ . Moreover, appealing to the definition of  $L$  in (14) and the convergence of the process  $c$  towards  $\eta$  as  $t \uparrow T$  in condition (10), we obviously have  $L_T = 0$  on the set  $\{\eta = 0\}$  as well as  $L_T > 0$  on the set  $\{0 < \eta < \infty\}$ . Concerning the “blow up” set  $\{\eta = +\infty\}$ , observe that we may write

$$L_t = c_0 e^{X_t - \frac{1}{2}[X]_t^c} \prod_{s \leq t} (1 + \Delta X_s) e^{-\Delta X_s} \quad (0 \leq t < T) \quad (17)$$

where  $X_t \triangleq -\int_0^t \frac{\nu_s}{c_{s-}} ds - \int_0^t \frac{1}{c_{s-}} dN_s$  (cf., e.g., Protter [21], Theorem II.37). Condition (12) guarantees  $\Delta X_s > -1$  for all  $0 \leq s < T$ . Moreover, applying Taylor’s formula, it holds for all  $0 \leq t < T$  that

$$\sum_{s \leq t} |\log((1 + \Delta X_s) e^{-\Delta X_s})| \leq \frac{1}{2} \int_{[0, T]} \frac{1}{c_{s-}^2} d[c]_s < +\infty \quad (18)$$

on the set  $\{\eta = +\infty\}$  by virtue of condition (13). This implies that the product of the jumps in (17) will converge to a strictly positive limit as  $t \uparrow T$  on  $\{\eta = +\infty\}$ . Concerning the limiting behaviour of the exponential  $\exp(X_t - \frac{1}{2}[X]_t^c)$  in (17) for  $t \uparrow T$ , observe that once more condition (13) prevents the limiting value from becoming 0 on  $\{\eta = +\infty\}$ . Indeed, the local martingale  $\int_0^t dN_s/c_{s-}$  cannot explode as  $t \uparrow T$  for those paths along which its quadratic variation  $\int_0^t d[c]_s/c_{s-}^2$  remains bounded on  $[0, T)$  (cf., e.g., Protter [21], Chapter V.2, for more details).  $\square$

Concerning the predetermined terminal target position  $\Xi_T$ , we will henceforth additionally assume that

$$\Xi_T L_T \in L^1(\mathbb{P}, \mathcal{F}_{T-}). \quad (19)$$

Now, we are in a position to introduce the key object of our approach:

**Definition 2.5.** For  $(\xi_t)_{0 \leq t \leq T}$  and  $\Xi_T$  satisfying (7) and (19), respectively, we define the *optimal signal process* as the càdlàg semimartingale on  $[0, T)$  given by

$$\hat{\xi}_t \triangleq \frac{1}{L_t} \mathbb{E} \left[ \Xi_T L_T + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right] \quad (0 \leq t < T). \quad (20)$$

Observe that  $\hat{\xi}$  can also be understood as the solution process to a linear BSDE with adjoint process  $L$  (cf., e.g., the book by Pham [18], Section 6.2.2, on the explicit solution to linear BSDEs in a Brownian framework). But the process  $\hat{\xi}$  may not possess a well defined terminal value in  $T$ . Indeed, by the definition in (20) we immediately observe that

$$\lim_{t \uparrow T} (\hat{\xi}_t L_t) = \Xi_T L_T \quad \mathbb{P}\text{-a.s.}$$

by the martingale convergence theorem. Hence, due to the convergence of the process  $L$  in Lemma 2.4, we can deduce that

$$\exists \lim_{t \uparrow T} \hat{\xi}_t = \Xi_T \quad \text{on the set } \{0 < \eta \leq +\infty\}. \quad (21)$$

In other words, the optimal signal process converges to the predetermined target position  $\Xi_T$  as  $t \uparrow T$  outside of the set  $\{\eta = 0\}$ . By contrast, on the set  $\{\eta = 0\}$ , which is not necessarily a  $\mathbb{P}$ -null set, we have  $L_T = 0$  by virtue of Lemma 2.4 and hence a limit of the process  $\hat{\xi}_t$  as  $t \uparrow T$  does not have to exist. As our analysis shows, though, the fact that the process  $\hat{\xi}$  may not possess a well defined terminal value in  $T$  is without harm.

*Remark 2.6* (Interpretation of the optimal signal). Let us present a way to interpret our optimal signal process  $\hat{\xi}$  defined in Definition 2.5. For ease of presentation and to avoid unnecessary technicalities, let us assume here that the convergence in (15) also holds in  $L^1(\mathbb{P})$  and that  $\nu \in L^1(\mathbb{P} \otimes dt)$  (these assumptions merely simplify the justification of the representation in (24) below; cf. Lemma 4.2 in Section 4). Further, since  $L_T > 0$  on the set  $\{0 < \eta \leq +\infty\}$  due to Lemma 2.4, note that  $\mathbb{E}[L_T] \neq 0$  (unless we are in the special case where  $\eta = 0$   $\mathbb{P}$ -a.s.). Then, by defining the *weight process*  $(w_t)_{0 \leq t < T}$  via

$$w_t \triangleq \frac{\mathbb{E}[L_T | \mathcal{F}_t]}{L_t} \quad (0 \leq t < T) \quad (22)$$

as well as the measure  $\mathbb{Q} \ll \mathbb{P}$  on  $(\Omega, \mathcal{F}_T)$  via

$$\frac{d\mathbb{Q}}{d\mathbb{P}} \triangleq \frac{L_T}{\mathbb{E}[L_T]},$$

we may write

$$\begin{aligned} \hat{\xi}_t &= \frac{1}{L_t} \mathbb{E} \left[ \Xi_T L_T + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right] \\ &= w_t \mathbb{E}_{\mathbb{Q}}[\Xi_T | \mathcal{F}_t] + (1 - w_t) \mathbb{E} \left[ \int_t^T \xi_r \frac{e^{-\int_t^r \frac{c_u}{\kappa_u} du}}{(1 - w_t) c_t} \nu_r dr \middle| \mathcal{F}_t \right] \end{aligned} \quad (23)$$

for all  $0 \leq t < T$ . Recall that the adjoint process  $(L_t)_{0 \leq t < T}$  is a strictly positive supermartingale by virtue of Lemma 2.4. Consequently, the weight process satisfies

$$0 \leq w_t < 1 \quad \mathbb{P}\text{-a.s. for all } 0 \leq t < T$$

(cf. Lemma 4.2 below for the strict right inequality). Moreover, we have the identity

$$\mathbb{E} \left[ \int_t^T \frac{e^{-\int_t^r \frac{c_u}{\kappa_u} du}}{(1 - w_t) c_t} \nu_r dr \middle| \mathcal{F}_t \right] = 1 \quad d\mathbb{P} \otimes dt\text{-a.e. on } \Omega \times [0, T) \quad (24)$$

(again, due to Lemma 4.2 below). That is, loosely speaking, the optimal signal process  $\hat{\xi}$  in (23) can be considered as a convex combination of a weighted average of expected future target positions of  $\xi$  and the expected

terminal position  $\Xi_T$ , computed under the auxiliary measure  $\mathbb{Q}$ , where the weight shifts gradually towards the ultimate terminal position  $\Xi_T$  as  $t \uparrow T$ , provided that  $\eta > 0$ . Indeed, by the definition of the weight process in (22), martingale convergence theorem and the convergence of the process  $L$  in Lemma 2.4, we have

$$\exists \lim_{t \uparrow T} w_t = 1 \quad \text{on the set } \{0 < \eta \leq +\infty\}.$$

Note that in the special case  $\eta = 0$   $\mathbb{P}$ -a.s. we would have  $L_T = 0$   $\mathbb{P}$ -a.s. due to Lemma 2.4 and hence the simpler representation

$$\hat{\xi}_t = \mathbb{E} \left[ \int_t^T \xi_r \frac{e^{-\int_t^r \frac{c_u}{\kappa_u} du}}{c_t} \nu_r dr \middle| \mathcal{F}_t \right] \quad (0 \leq t < T)$$

with the same property as in (24) with  $w \equiv 0$  (again, see Lemma 4.2 below).

## 2.2 Objective functional

We return to our stochastic LQ problem with singular stochastic terminal state constraint introduced in our introduction. Recall that for given  $x \in \mathbb{R}$  we want to find a progressively measurable control  $u \in L^1([0, T], ds)$   $\mathbb{P}$ -a.s. with controlled process

$$X_t^u \triangleq x + \int_0^t u_s ds \quad (0 \leq t \leq T) \quad (25)$$

which minimizes

$$\mathbb{E} \left[ \int_0^T (X_t^u - \xi_t)^2 \nu_t dt + \int_0^T \kappa_t u_t^2 dt + \eta (X_T^u - \Xi_T)^2 \right]. \quad (26)$$

The delicate issue here is that we allow the random penalization parameter  $\eta$  to take the value  $+\infty$  with positive but not necessarily full probability in order to incorporate the stochastic terminal state constraint  $X_T^u = \Xi_T$  on  $\{\eta = +\infty\}$ . As a result, the cost term  $\mathbb{E}[\eta(X_T^u - \Xi_T)^2]$  needs to be defined in a suitable way.

One possible approach in order to tackle the random singularity at terminal time  $T$  consists of performing a *truncation in space*, that is, looking at a family of unconstrained variants of the problem in (26) where the random

penalization parameter  $\eta$  is replaced by truncated versions  $\eta \wedge n$  for some constants  $n > 0$ . After having solved these auxiliary problems, one can try to pass to the limit  $n \uparrow \infty$ . In the very specific case  $\xi \equiv \Xi_T = 0$ , this has been done in Kruse and Popier [15]. Recall that the stochastic LQ problem in (26) with bounded penalization parameter  $\eta \wedge n$  (as well as bounded processes  $\kappa$  and  $\nu$ ) has been solved in Kohlmann and Tang [14] (within a Brownian framework) and is fully characterized by the two BS(R)DEs in (3) and (4) with terminal conditions  $c_T = \eta \wedge n$  and  $b_T = (\eta \wedge n)\Xi_T$ , respectively. As shown by, e.g., Kruse and Popier [15] and Graewe et al. [11], one can indeed let  $n \uparrow \infty$  in the BSRDE for  $c$ . When  $\Xi_T \neq 0$ , though, this is not possible for the linear BSDE in (4) in general.

Our main idea to tackle and resolve the delicate stochastic terminal state constraint consists of performing a *truncation in time* instead of space. Specifically, we propose to define the problem in (26) as a properly chosen limit of stochastic LQ problems with terminal times  $\tau < T$ . The delicate final state penalty  $\eta(X_T^u - \Xi_T)^2$  is replaced by an appropriate penalization term at each time  $\tau$ . Of course, it is natural to replace  $\eta$  with  $c_\tau$ , but not clear at all what should replace  $\Xi_T$  in order to get time consistent penalization terms. As it turns out, the optimal signal process  $\hat{\xi}$  at time  $\tau$  gives such a canonical replacement. That is, in light of  $\lim_{t \uparrow T} \hat{\xi}_t = \Xi_T$  on  $\{0 < \eta \leq +\infty\}$  in (21) as well as  $\lim_{t \uparrow T} c_t = \eta$  in (10), we propose to define the performance functional as follows:

$$J(u) \triangleq \limsup_{\tau \uparrow T} \mathbb{E} \left[ \int_0^\tau (X_t^u - \xi_t)^2 \nu_t dt + \int_0^\tau \kappa_t u_t^2 dt + c_\tau (X_\tau^u - \hat{\xi}_\tau)^2 \right]. \quad (27)$$

The limes superior is taken over all sequences of stopping times  $(\tau^n)_{n=1,2,\dots}$  which  $\mathbb{P}$ -a.s. converge strictly from below to the terminal time  $T$ . The set of admissible controls is defined to be the domain of  $J$ :

$$\mathcal{U} \triangleq \{u \in L^1(dt) \text{ } \mathbb{P}\text{-a.s. progressively measurable with } J(u) < +\infty\}. \quad (28)$$

Note that, appealing to Fatou's Lemma as well as (10) and (21), all controls  $u \in \mathcal{U}$  necessarily satisfy the random terminal state constraint

$$X_T^u = \Xi_T \quad \text{on the set } \{\eta = +\infty\}.$$

The optimization problem we want to solve can now be formulated as

$$J(u) \rightarrow \min_{u \in \mathcal{U}}. \quad (29)$$

### 3 Main result

We are now ready to state our main theorem. As it turns out, the optimal control to our stochastic LQ problem in (29) with singular stochastic terminal state constraint and its corresponding optimal value are fully characterized by the processes  $c$  and  $\hat{\xi}$ . First of all, we have to ensure that our set of admissible controls defined in (28) is not empty. In fact, it follows from our analysis below that  $\mathcal{U} \neq \emptyset$  if and only if

$$\mathbb{E} \left[ \int_0^T (\xi_t - \hat{\xi}_t)^2 \nu_t dt \right] < +\infty \quad \text{and} \quad \mathbb{E} \left[ \int_{[0,T)} c_t d[\hat{\xi}]_t \right] < +\infty, \quad (30)$$

where  $[\hat{\xi}]$  denotes the quadratic variation process of the semimartingale  $\hat{\xi}$ . In particular, (30) are necessary and sufficient for well posedness of (29).

**Theorem 3.1.** *Let Assumption 2.2 as well as conditions (6), (7), (8) and (19) hold true. Then, we have  $\mathcal{U} \neq \emptyset$  if and only if (30) is satisfied. In this case, the optimal control  $\hat{u} \in \mathcal{U}$  for problem (29) with controlled process  $\hat{X} \triangleq X^{\hat{u}}$  is given by the feedback law*

$$\hat{u}_t = \frac{c_t}{\kappa_t} (\hat{\xi}_t - \hat{X}_t) \quad (0 \leq t < T), \quad (31)$$

and the minimal costs are

$$J(\hat{u}) = c_0(x - \hat{\xi}_0)^2 + \mathbb{E} \left[ \int_0^T (\xi_t - \hat{\xi}_t)^2 \nu_t dt \right] + \mathbb{E} \left[ \int_{[0,T)} c_t d[\hat{\xi}]_t \right]. \quad (32)$$

The proof of Theorem 3.1 is deferred to Section 4 below. Observe that the feedback law of the optimal control in (31) prescribes a reversion towards the optimal signal process  $\hat{\xi}_t$  rather than towards the current target position  $\xi_t$ . The reversion speed is controlled by the ratio  $c/\kappa$ . In particular, on the “blow-up” set  $\{\eta = +\infty\}$  the optimizer reverts with increased urgency towards the optimal signal  $\hat{\xi}$  and hence to the ultimate target position  $\Xi_T$  due to (21). This result generalizes the insights from the constant coefficient case with almost sure terminal state constraint which are presented in Bank et al. [4].

Due to the integrability conditions in (30) the optimal costs  $J(\hat{u})$  in (32) of the optimizer  $\hat{u}$  in (31) are obviously finite. Actually, they nicely separate into three intuitively appealing terms making transparent how the regularity and predictability of the targets  $\xi$  and  $\Xi_T$  determine the problem’s optimal

value. The first term represents the costs due to a possibly suboptimal initial position  $x$ . The second term shows how the regularity of the target process  $\xi$  feeds into the overall costs: Targets which are poorly approximated by the optimal signal process  $\hat{\xi}$  in the  $L^2(\mathbb{P} \otimes \nu_t dt)$ -sense produce higher costs. Finally, the third term reveals the importance of the optimal signal's quadratic variation process  $[\hat{\xi}]$ . Referring to the definition of  $\hat{\xi}$  in (20) (cf. also the representation in (23)), the quadratic variation  $[\hat{\xi}]$  can be viewed as a measure for the strength of the fluctuations in the assessment of the average future target positions of  $\xi$ , the terminal position  $\Xi_T$  and the random variable  $L_T$  which involves the outcome of the penalization parameter  $\eta$  at time  $T$ . With this respect, the second integrability condition in (30) can be interpreted as encoding a condition on the predictability of the final stochastic target position  $\Xi_T$  as well as the random penalization parameter  $\eta$ . Loosely speaking, it ensures that the outcome of the final position  $\Xi_T$  as well as the “blow-up” event  $\{\eta = +\infty\}$  on which  $\Xi_T$  has to be matched by controls in  $\mathcal{U}$  are not allowed to come as “too big a surprise” at final time  $T$ . A similar condition is also formulated in Bank et al. [4], Remark 2.1 and Lemma 5.4, in the case of constant coefficients and almost sure terminal state constraint. Ankirchner and Kruse [2] confine themselves to stochastic terminal state constraints of the form  $\Xi_T = \int_0^T \lambda_t dt$  for some progressively measurable and suitably integrable process  $(\lambda_t)_{0 \leq t \leq T}$  which are gradually revealed as  $t \uparrow T$ .

*Remark 3.2* (Related results in the literature). 1. In the case of constant coefficients  $\nu_t \equiv \nu \in \mathbb{R}_+$ ,  $\kappa_t \equiv \kappa \in \mathbb{R}_+$  and  $\eta \in [0, +\infty]$  the BSRDE in (9) boils down to a deterministic *ordinary Riccati differential equation* on  $[0, T]$  of the form

$$c'_t = \frac{c_t^2}{\kappa} - \nu \quad \text{subject to } c_T = \eta$$

with explicitly available solutions

$$c_t = \begin{cases} \sqrt{\nu\kappa} \frac{\sqrt{\nu\kappa} \sinh\left(\frac{\sqrt{\nu/\kappa}(T-t)}{\sqrt{\nu\kappa}}\right) + \eta \cosh\left(\frac{\sqrt{\nu/\kappa}(T-t)}{\sqrt{\nu\kappa}}\right)}{\eta \sinh\left(\frac{\sqrt{\nu/\kappa}(T-t)}{\sqrt{\nu\kappa}}\right) + \sqrt{\nu\kappa} \cosh\left(\frac{\sqrt{\nu/\kappa}(T-t)}{\sqrt{\nu\kappa}}\right)} & 0 \leq \eta < +\infty, \\ \sqrt{\nu\kappa} \coth\left(\sqrt{\nu}(T-t)/\sqrt{\kappa}\right), & \eta = +\infty \end{cases},$$

for all  $0 \leq t \leq T$ . Consequently, the process  $L$  given in (14) is also just deterministic and the optimal signal process  $\hat{\xi}$  in (20) can be computed explicitly (up to the conditional expectation). In particular, we recover the explicit results from Bank et al. [4]. Therein, the first integrability

condition in (30) holds true as soon as  $\xi \in L^2(\mathbb{P} \otimes dt)$  and the second condition is replaced by a condition on the terminal position  $\Xi_T$  (cf. Remark 2.1 and Lemma 5.4 in [4]).

2. In the special case  $\xi \equiv 0$  and  $\Xi_T = 0$   $\mathbb{P}$ -a.s., we recover under the specific dynamics of the controlled state process  $X^u$  in (25) the result obtained in Kruse and Popier [15], Theorem 3. In this setup, the optimal control  $u^0$  with controlled process  $X^0$  is given by

$$u_t^0 = -\frac{c_t}{\kappa_t} X_t^0 = -\frac{c_t}{\kappa_t} x e^{-\int_0^t \frac{c_u}{\kappa_u} du} = -\frac{x L_t}{\kappa_t} \quad \text{for all } 0 \leq t \leq T.$$

Observe that the adjoint process  $(L_t)_{0 \leq t \leq T}$  defined in (14) is linked to the optimal control  $u^0$  via the relation  $L = -\kappa u^0$  (if we set  $x = 1$ ). The corresponding optimal costs are given by

$$J(u^0) = c_0 x^2.$$

In fact, Kruse and Popier [15] show that the process  $(x^2 c_t)_{0 \leq t < T}$  with  $(c_t)_{0 \leq t < T}$  satisfying the BSRDE in (9) (with a slightly weaker singular terminal condition, recall Remark 2.3 3.)) is the *value process* to this particular optimization problem (cf. also the Feynman-Kac representation result in Kohlmann and Tang [14], Section 3.5, of the solution process to the BSRDE in (9) with bounded  $\eta$ ). From a Mathematical Finance point of view, the process  $(c_t)_{0 \leq t < T}$  can therefore be regarded as the *optimal liquidation cost process* in the sense that  $c_t$  provides the minimal costs at time  $t$  with respect to the remaining time to maturity  $T - t$  of unwinding one unit of current asset holdings ( $x = 1$ ) until  $T$  if the event  $\{\eta = +\infty\}$  occurs. The process  $L$  is characterized by the corresponding *optimal liquidation rate*  $u^0$  and the price impact process  $\kappa$ . In particular, the terminal value  $L_T$  represents, with reversed sign, the ultimate optimal liquidation rate  $u_T^0$  weighted with the ultimate instantaneous price impact  $\kappa_T$ .

3. In the case where  $\eta$  is bounded  $\mathbb{P}$ -a.s. as well as the processes  $(\nu_t)_{0 \leq t \leq T}$  and  $(\kappa_t)_{0 \leq t \leq T}$ , we recover, under the specific dynamics of the controlled state process  $X^u$  in (25), the results obtained in Kohlmann and Tang [14], Theorem 5.2, established within a Brownian framework. Note, however, that they characterize the optimal control  $\hat{u}$  in terms of the process  $c$  and the solution process to the linear BSDE in (4) which does



not correspond to  $\hat{\xi}$ . In particular, the key role played by the process  $\hat{\xi}$  was not observed in [14].

## 4 Proofs

Throughout this section we work under the assumptions of our main result, Theorem 3.1. Its verification relies on a completion of squares argument similar to Kohlmann and Tang [14] (cf. also Yong and Zhou [24] for this method in solving LQ problems). The following lemma summarizes the key identity for our verification and illustrates again the usefulness of our signal process  $\hat{\xi}$ .

**Lemma 4.1.** *For all progressively measurable,  $\mathbb{P}$ -a.s. locally  $L^1([0, T], dt)$ -integrable processes  $u$ , the cost process*

$$C_t(u) \triangleq \int_0^t (X_s^u - \xi_s)^2 \nu_s ds + \int_0^t \kappa_s u_s^2 ds + c_t (X_t^u - \hat{\xi}_t)^2 \quad (0 \leq t < T) \quad (33)$$

is a nonnegative, càdlàg local submartingale. It allows for the decomposition

$$C_t(u) = c_0(x - \hat{\xi}_0)^2 + A_t(u) + M_t(u) \quad (0 \leq t < T) \quad (34)$$

for all  $0 \leq t < T$ , where

$$\begin{aligned} A_t(u) \triangleq & \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds + \int_0^t c_s d[\hat{\xi}]_s \\ & + \int_0^t \kappa_s \left( u_s - \frac{c_s}{\kappa_s} (\hat{\xi}_s - X_s^u) \right)^2 ds \end{aligned} \quad (35)$$

is a right continuous, nondecreasing, adapted process and

$$M_t(u) \triangleq \int_0^t (\hat{\xi}_{s-}^2 - (X_{s-}^u)^2) dN_s + 2 \int_0^t \frac{c_{s-}}{L_{s-}} (\hat{\xi}_{s-} - X_{s-}^u) d\tilde{M}_s \quad (36)$$

with

$$\tilde{M}_t \triangleq \mathbb{E} \left[ \Xi_T L_T + \int_0^T \xi_t e^{-\int_0^t \frac{c_u}{\kappa_u} du} \nu_t dt \middle| \mathcal{F}_t \right] \quad (37)$$

is a local martingale on  $[0, T)$ .

*Proof.* Let us expand

$$c_t(X_t^u - \hat{\xi}_t)^2 = c_t(X_t^u)^2 - 2X_t^u c_t \hat{\xi}_t + c_t \hat{\xi}_t^2 \quad (0 \leq t < T)$$

and then apply Itô's formula to each of the resulting three terms. This will be prepared by computing the dynamics of the processes  $\hat{\xi}$ ,  $c\hat{\xi}$  and  $c\hat{\xi}^2$ , respectively, in the following steps 1, 2 and 3. In step 4 we put everything together and derive our main identity (33).

*Step 1:* We start with computing the dynamics of our optimal signal process  $(\hat{\xi}_t)_{0 \leq t < T}$  defined in (20). For ease of notation, let us define the process

$$Y_t \triangleq \int_0^t \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \quad (0 \leq t \leq T).$$

Observe that  $Y_T \in L^1(\mathbb{P})$  due to (7). Moreover, recall that  $\Xi_T L_T \in L^1(\mathcal{F}_{T-}, \mathbb{P})$  by (19) so that (37) defines a càdlàg martingale on  $[0, T]$ . By the definition of  $\hat{\xi}$  in (20), we can now express  $\hat{\xi}$  in terms of  $Y$  and  $\tilde{M}$  via

$$\hat{\xi}_t = \frac{1}{L_t} \mathbb{E} \left[ \Xi_T L_T + \int_t^T \xi_r e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right] = \frac{1}{L_t} (\tilde{M}_t - Y_t) \quad (38)$$

for all  $0 \leq t < T$ . Next, recall the dynamics of  $L$  on  $[0, T)$  in (16) and note that

$$\Delta L_t = -\frac{L_{t-}}{c_{t-}} \Delta N_t \quad \text{and} \quad [L]_t^c = \int_0^t \frac{L_{s-}^2}{c_{s-}^2} d[N]_s^c, \quad (39)$$

where  $[L]^c$  and  $[N]^c$  denote the path-by-path continuous parts of the quadratic variations of  $[L]$  and  $[N]$ , respectively (cf., e.g., Protter [21], Chapter II.6, for more details). Hence, applying Itô's formula as in, e.g., [21], Theorem II.32, we obtain

$$\begin{aligned} \frac{1}{L_t} &= \frac{1}{L_0} - \int_0^t \frac{1}{L_{s-}^2} dL_s + \int_0^t \frac{1}{L_{s-}^3} d[L]_s^c \\ &\quad + \sum_{s \leq t} \left( \frac{1}{L_s} - \frac{1}{L_{s-}} + \frac{1}{L_{s-}^2} \Delta L_s \right). \end{aligned} \quad (40)$$

Using (39), the summands in the sum in (40) above can be written as

$$\frac{L_{s-} - L_s}{L_s L_{s-}} - \frac{\Delta N_s}{L_{s-} c_{s-}} = \frac{\Delta N_s}{c_{s-}} \frac{L_{s-} - L_s}{L_s L_{s-}} = \frac{(\Delta N_s)^2}{L_s c_{s-}^2} = \frac{(\Delta N_s)^2}{L_{s-} c_{s-} c_s},$$

where we also used  $\Delta c_s = -\Delta N_s$  and thus the identity  $1/L_s = c_{s-}/(L_s - c_s)$  in the last equality. Hence, together with the dynamics of  $L$  in (16) and  $[L]^c$  in (39) we can rewrite (40) as

$$\begin{aligned} \frac{1}{L_t} &= \frac{1}{L_0} + \int_0^t \frac{\nu_s}{L_s - c_{s-}} ds + \int_0^t \frac{1}{L_s - c_{s-}} dN_s \\ &\quad + \int_0^t \frac{1}{L_s - c_{s-}^2} d[N]_s^c + \sum_{s \leq t} \frac{(\Delta N_s)^2}{L_s - c_{s-} - c_s}. \end{aligned} \quad (41)$$

Now, integrating by parts in (38) and then using the dynamics of  $1/L$  in (41) gives us

$$\begin{aligned} \hat{\xi}_t &= \hat{\xi}_0 + \int_0^t \frac{1}{L_{s-}} (d\tilde{M}_s - dY_s) + \int_0^t \hat{\xi}_{s-} L_{s-} d\left(\frac{1}{L_s}\right) + \left[\frac{1}{L}, \tilde{M}\right]_t \\ &= \hat{\xi}_0 - \int_0^t (\xi_s - \hat{\xi}_{s-}) \frac{\nu_s}{c_{s-}} ds + \int_0^t \frac{1}{L_{s-}} d\tilde{M}_s + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}} dN_s \\ &\quad + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}^2} d[N]_s^c + \sum_{s \leq t} \frac{\hat{\xi}_{s-}}{c_{s-} - c_s} (\Delta N_s)^2 + \left[\frac{1}{L}, \tilde{M}\right]_t, \end{aligned} \quad (42)$$

where the quadratic covariation can be computed as

$$\begin{aligned} \left[\frac{1}{L}, \tilde{M}\right]_t &= \int_0^t \frac{1}{L_s - c_{s-}} d[\tilde{M}, N]_s^c \\ &\quad + \sum_{s \leq t} \left( \frac{\Delta \tilde{M}_s \Delta N_s}{L_s - c_{s-}} + \frac{(\Delta N_s)^2 \Delta \tilde{M}_s^2}{L_s - c_{s-} - c_s} \right). \end{aligned} \quad (43)$$

Collecting all the sums in (42) together with those in (43) yields

$$\begin{aligned} &\sum_{s \leq t} \frac{\Delta N_s}{L_s - c_{s-} - c_s} \left( c_s \Delta \tilde{M}_s + \Delta N_s \Delta \tilde{M}_s + \hat{\xi}_{s-} L_{s-} \Delta N_s \right) \\ &= \sum_{s \leq t} \frac{\Delta N_s}{L_s - c_{s-} - c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right), \end{aligned} \quad (44)$$

where we used the fact that

$$\Delta \tilde{M}_s = \tilde{M}_s - \tilde{M}_{s-} = \hat{\xi}_s L_s - \hat{\xi}_{s-} L_{s-} \quad (45)$$

due to the representation of  $\hat{\xi}$  in (38) and the continuity of  $Y$ . Plugging back (44) into (42) finally gives us

$$\begin{aligned}\hat{\xi}_t &= \hat{\xi}_0 - \int_0^t (\xi_s - \hat{\xi}_{s-}) \frac{\nu_s}{c_{s-}} ds + \int_0^t \frac{1}{L_{s-}} d\tilde{M}_s + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}} dN_s \\ &\quad + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}^2} d[N]_s^c + \int_0^t \frac{1}{L_{s-}c_{s-}} d[\tilde{M}, N]_s^c \\ &\quad + \sum_{s \leq t} \frac{\Delta N_s}{L_{s-}c_{s-}c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right).\end{aligned}\tag{46}$$

*Step 2:* Let us now compute the dynamics of  $c\hat{\xi}$ . Again, integration by parts, together with the dynamics of  $\hat{\xi}$  in (46), yields

$$\begin{aligned}c_t \hat{\xi}_t &= c_0 \hat{\xi}_0 + \int_0^t c_{s-} d\hat{\xi}_s + \int_0^t \hat{\xi}_{s-} dc_s + [c, \hat{\xi}]_t \\ &= c_0 \hat{\xi}_0 - \int_0^t \xi_s \nu_s ds + \int_0^t \hat{\xi}_{s-} \frac{c_s^2}{\kappa_s} ds + \int_0^t \frac{c_{s-}}{L_{s-}} d\tilde{M}_s \\ &\quad + \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}} d[N]_s^c + \int_0^t \frac{1}{L_{s-}} d[\tilde{M}, N]_s^c \\ &\quad + \sum_{s \leq t} \frac{\Delta N_s}{L_{s-}c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right) + [c, \hat{\xi}]_t.\end{aligned}\tag{47}$$

The quadratic covariation in (47) can be computed as

$$\begin{aligned}[c, \hat{\xi}]_t &= - \int_0^t \frac{1}{L_{s-}} d[\tilde{M}, N]_s^c - \int_0^t \frac{\hat{\xi}_{s-}}{c_{s-}} d[N]_s^c \\ &\quad - \sum_{s \leq t} \frac{\Delta N_s \Delta \tilde{M}_s}{L_{s-}} - \sum_{s \leq t} \frac{\hat{\xi}_{s-} (\Delta N_s)^2}{c_{s-}} \\ &\quad - \sum_{s \leq t} \frac{(\Delta N_s)^2}{L_{s-}c_{s-}c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right).\end{aligned}\tag{48}$$

The sums of the jumps in the quadratic covariation in (48) can be rewritten

(using again the identity in (45) as well as the fact that  $\Delta c_s = -\Delta N_s$ ) as

$$\begin{aligned} & - \sum_{s \leq t} \frac{\Delta N_s}{L_{s-} c_{s-} c_s} \left( \Delta \tilde{M}_s c_s c_{s-} + \hat{\xi}_{s-} \Delta N_s L_{s-} c_s + \Delta N_s \hat{\xi}_s L_s c_{s-} - \Delta N_s \hat{\xi}_{s-} L_{s-} c_s \right) \\ & = - \sum_{s \leq t} \frac{\Delta N_s}{L_{s-} c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right). \end{aligned}$$

With this observation, plugging back the quadratic covariation in (48) into (47), we simply get

$$c_t \hat{\xi}_t = c_0 \hat{\xi}_0 - \int_0^t \xi_s \nu_s ds + \int_0^t \hat{\xi}_{s-} \frac{c_s^2}{\kappa_s} ds + \int_0^t \frac{c_{s-}}{L_{s-}} d\tilde{M}_s. \quad (49)$$

*Step 3:* Next, we compute the dynamics of  $c \hat{\xi}^2$ . Application of integration by parts together with the dynamics of  $\hat{\xi}$  in (46) yields

$$\begin{aligned} \hat{\xi}_t^2 &= \hat{\xi}_0^2 + 2 \int_0^t \hat{\xi}_{s-} d\hat{\xi}_s + [\hat{\xi}]_t \\ &= \hat{\xi}_0^2 - 2 \int_0^t \hat{\xi}_{s-} (\xi_s - \hat{\xi}_{s-}) \frac{\nu_s}{c_{s-}} ds + 2 \int_0^t \frac{\hat{\xi}_{s-}}{L_{s-}} d\tilde{M}_s + 2 \int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}} dN_s \\ &\quad + 2 \int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}^2} d[N]_s^c + 2 \int_0^t \frac{\hat{\xi}_{s-}}{L_{s-} c_{s-}} d[\tilde{M}, N]_s^c \\ &\quad + 2 \sum_{s \leq t} \frac{\hat{\xi}_{s-} \Delta N_s}{L_{s-} c_{s-} c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right) + [\hat{\xi}]_t. \end{aligned}$$

Consequently, using once more integration by parts, we obtain

$$\begin{aligned} c_t \hat{\xi}_t^2 &= c_0 \hat{\xi}_0^2 + \int_0^t c_{s-} d\hat{\xi}_s^2 + \int_0^t \hat{\xi}_{s-}^2 dc_s + [c, \hat{\xi}^2]_t \\ &= c_0 \hat{\xi}_0^2 - 2 \int_0^t \hat{\xi}_{s-} (\xi_s - \hat{\xi}_{s-}) \nu_s ds + 2 \int_0^t \frac{c_{s-} \hat{\xi}_{s-}}{L_{s-}} d\tilde{M}_s + 2 \int_0^t \hat{\xi}_{s-}^2 dN_s \\ &\quad + 2 \int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}} d[N]_s^c + 2 \int_0^t \frac{\hat{\xi}_{s-}}{L_{s-}} d[\tilde{M}, N]_s^c \\ &\quad + 2 \sum_{s \leq t} \frac{\hat{\xi}_{s-} \Delta N_s}{L_{s-} c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right) + \int_0^t c_{s-} d[\hat{\xi}]_s \\ &\quad + \int_0^t \hat{\xi}_{s-}^2 \frac{c_s^2}{\kappa_s} ds - \int_0^t \hat{\xi}_{s-}^2 \nu_s ds - \int_0^t \hat{\xi}_{s-}^2 dN_s + [c, \hat{\xi}^2]_t. \end{aligned} \quad (50)$$

The final quadratic covariation in (50) can be computed as

$$\begin{aligned}
[c, \hat{\xi}^2]_t &= -2 \int_0^t \frac{\hat{\xi}_{s-}}{L_{s-}} d[\tilde{M}, N]_s^c - 2 \sum_{s \leq t} \frac{\hat{\xi}_{s-}}{L_{s-}} \Delta \tilde{M}_s \Delta N_s \\
&\quad - 2 \int_0^t \frac{\hat{\xi}_{s-}^2}{c_{s-}} d[N]_s^c - 2 \sum_{s \leq t} \frac{\hat{\xi}_{s-}^2}{c_{s-}} (\Delta N_s)^2 \\
&\quad - 2 \sum_{s \leq t} \frac{\hat{\xi}_{s-} (\Delta N_s)^2}{L_{s-} c_{s-} c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right) + \int_0^t \Delta c_s d[\hat{\xi}]_s. \quad (51)
\end{aligned}$$

Observe that the sum of jumps in (51) can be rewritten as

$$\begin{aligned}
&-2 \sum_{s \leq t} \frac{\hat{\xi}_{s-} \Delta N_s}{L_{s-} c_{s-} c_s} \left( \Delta \tilde{M}_s c_s c_{s-} + \hat{\xi}_{s-} \Delta N_s c_s L_{s-} \right. \\
&\quad \left. + \Delta N_s \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \Delta N_s \right) \\
&= -2 \sum_{s \leq t} \frac{\hat{\xi}_{s-} \Delta N_s}{L_{s-} c_s} \left( \hat{\xi}_s L_s c_{s-} - \hat{\xi}_{s-} L_{s-} c_s \right),
\end{aligned}$$

where we used once more the identity in (45) and  $\Delta c_s = -\Delta N_s$ . With this observation, plugging back (51) into (50), we finally obtain

$$\begin{aligned}
c_t \hat{\xi}_t^2 &= c_0 \hat{\xi}_0^2 - 2 \int_0^t \hat{\xi}_{s-} \xi_s \nu_s ds + \int_0^t \hat{\xi}_{s-}^2 \nu_s ds + 2 \int_0^t \frac{c_{s-} \hat{\xi}_{s-}}{L_{s-}} d\tilde{M}_s \\
&\quad + \int_0^t \hat{\xi}_{s-}^2 dN_s + \int_0^t c_s d[\hat{\xi}]_s + \int_0^t \hat{\xi}_{s-}^2 \frac{c_s^2}{\kappa_s} ds. \quad (52)
\end{aligned}$$

*Step 4:* Let us now put together all the computations from the preceding steps. Specifically, let  $u$  be a progressively measurable,  $\mathbb{P}$ -a.s. locally  $L^1([0, T], dt)$ -integrable process with corresponding controlled process  $X^u$ . Due to our computations in (49) and (52) as well as the fact that  $X^u$  is

continuous and of finite variation, we get for all  $0 \leq t < T$  that

$$\begin{aligned}
c_t(X_t^u - \hat{\xi}_t)^2 &= c_t(X_t^u)^2 - 2X_t^u c_t \hat{\xi}_t + c_t \hat{\xi}_t^2 \\
&= c_0(x - \hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s - \int_0^t (X_s^u)^2 \nu_s ds + 2 \int_0^t X_s^u \nu_s \hat{\xi}_s ds \\
&\quad - 2 \int_0^t c_s u_s (\hat{\xi}_s - X_s^u) ds + \int_0^t \frac{c_s^2}{\kappa_s} (X_s^u - \hat{\xi}_s)^2 ds - 2 \int_0^t \hat{\xi}_s \xi_s \nu_s ds + \int_0^t \hat{\xi}_s^2 \nu_s ds \\
&\quad + \int_0^t (\hat{\xi}_{s-}^2 - (X_{s-}^u)^2) dN_s + 2 \int_0^t \frac{c_{s-}}{L_{s-}} (\hat{\xi}_{s-} - X_{s-}^u) d\tilde{M}_s. \tag{53}
\end{aligned}$$

Observe that the last two stochastic integrands sum up to  $M_t(u)$  defined in (36). Furthermore, two completions of squares in the third line of (53) yield

$$\begin{aligned}
c_t(X_t^u - \hat{\xi}_t)^2 &= c_0(x - \hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s - \int_0^t (X_s^u)^2 \nu_s ds + 2 \int_0^t X_s^u \nu_s \hat{\xi}_s ds \\
&\quad + \int_0^t \kappa_s \left( u_s - \frac{c_s}{\kappa_s} (\hat{\xi}_s - X_s^u) \right)^2 ds + \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds \\
&\quad - \int_0^t \kappa_s u_s^2 ds - \int_0^t \xi_s^2 \nu_s ds + M_t(u) \\
&= c_0(x - \hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s - \int_0^t (X_s^u - \xi_s)^2 \nu_s ds \\
&\quad + \int_0^t \kappa_s \left( u_s - \frac{c_s}{\kappa_s} (\hat{\xi}_s - X_s^u) \right)^2 ds + \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds \\
&\quad - \int_0^t \kappa_s u_s^2 ds + M_t(u)
\end{aligned}$$

Consequently, we can write

$$\begin{aligned}
0 \leq C_t(u) &= \int_0^t (X_s^u - \xi_s)^2 \nu_s ds + \int_0^t \kappa_s u_s^2 ds + c_t(X_t^u - \hat{\xi}_t)^2 \\
&= c_0(x - \hat{\xi}_0)^2 + \int_0^t c_s d[\hat{\xi}]_s + \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds \\
&\quad + \int_0^t \kappa_s \left( u_s - \frac{c_s}{\kappa_s} (\hat{\xi}_s - X_s^u) \right)^2 ds + M_t(u) \\
&= c_0(x - \hat{\xi}_0)^2 + A_t(u) + M_t(u) \quad (0 \leq t < T) \tag{54}
\end{aligned}$$

with  $(A_t(u))_{0 \leq t < T}$  as defined in (35). Finally, observe that the process  $(A_t(u))_{0 \leq t < T}$  is a right continuous, nondecreasing, adapted process and that  $(M_t(u))_{0 \leq t < T}$  is a càdlàg local martingale because  $\tilde{M}$  and  $N$  are local martingales on  $[0, T)$  and all integrands in (36) are left continuous (cf., e.g., Protter [21], Theorem III.33). Consequently, we have that  $(C_t(u))_{0 \leq t < T}$  is a nonnegative, càdlàg local submartingale.  $\square$

We are now ready to give the proof of our main Theorem 3.1:

**Proof of Theorem 3.1:** First, let us assume that  $\mathcal{U} \neq \emptyset$ . For any  $u \in \mathcal{U}$  we can consider the corresponding cost process  $C_t(u) = c_0(x - \hat{\xi}_0)^2 + A_t(u) + M_t(u)$ ,  $0 \leq t < T$ , as in (34) in Lemma 4.1 above. Let  $(\tau^n)_{n=1,2,\dots}$  be a localizing sequence of stopping times of the local martingale  $(M_t(u))_{0 \leq t < T}$  such that  $\tau^n \uparrow T$   $\mathbb{P}$ -a.s. strictly from below as  $n \rightarrow \infty$  and  $(M_{t \wedge \tau^n}(u))_{0 \leq t < T}$  is a uniformly integrable martingale for each  $n$  (cf., e.g., Protter [21], Chapter I.6, for more details). Then it holds by definition of our performance functional  $J$  in (27) that

$$\begin{aligned}
\infty > J(u) &\triangleq \limsup_{\tau \uparrow T} \mathbb{E}[C_\tau(u)] \\
&\geq c_0(x - \hat{\xi}_0)^2 + \limsup_{n \rightarrow \infty} \{\mathbb{E}[A_{\tau^n}(u)] + \mathbb{E}[M_{\tau^n}(u)]\} \\
&= c_0(x - \hat{\xi}_0)^2 \\
&\quad + \limsup_{n \rightarrow \infty} \left\{ \mathbb{E} \left[ \int_0^{\tau^n} (\xi_s - \hat{\xi}_s)^2 \nu_s ds + \int_0^{\tau^n} c_s d[\hat{\xi}]_s \right. \right. \\
&\quad \quad \left. \left. + \int_0^{\tau^n} \kappa_s \left( u_s - \frac{c_s}{\kappa_s} (\hat{\xi}_s - X_s^u) \right)^2 ds \right] \right\} \\
&\geq c_0(x - \hat{\xi}_0)^2 \\
&\quad + \limsup_{n \rightarrow \infty} \left\{ \mathbb{E} \left[ \int_0^{\tau^n} (\xi_s - \hat{\xi}_s)^2 \nu_s ds + \int_0^{\tau^n} c_s d[\hat{\xi}]_s \right] \right\} \\
&= c_0(x - \hat{\xi}_0)^2 + \mathbb{E} \left[ \int_0^T (\xi_s - \hat{\xi}_s)^2 \nu_s ds \right] + \mathbb{E} \left[ \int_{[0,T)} c_s d[\hat{\xi}]_s \right], \quad (55)
\end{aligned}$$

where we used Doob's Optional Sampling Theorem as, e.g., in Protter [21], Theorem I.16, in order to get  $\mathbb{E}[M_{\tau^n}(u)] = 0$  for all  $n \geq 1$ , and the last equality is due to monotone convergence. In particular, the computations in



(55) show that (30) necessarily holds true. In other words, setting

$$v \triangleq c_0(x - \hat{\xi}_0)^2 + \mathbb{E} \left[ \int_0^T (\xi_s - \hat{\xi}_s)^2 \nu_s ds \right] + \mathbb{E} \left[ \int_{[0,T)} c_s d[\hat{\xi}]_s \right] < \infty, \quad (56)$$

we have for all  $u \in \mathcal{U}$  the lower bound

$$J(u) \geq v. \quad (57)$$

Now, let us define the control  $\hat{u}$  with corresponding controlled process  $\hat{X} \triangleq X^{\hat{u}}$  via the feedback law

$$\hat{u}_t = \frac{c_t}{\kappa_t} (\hat{\xi}_t - \hat{X}_t) \quad (0 \leq t < T).$$

Observe that  $\hat{u}$  is a progressively measurable process and  $\mathbb{P}$ -a.s. locally  $L^1([0, T], dt)$ -integrable. We denote by  $C_t(\hat{u}) = c_0(x - \hat{\xi}_0)^2 + M_t(\hat{u}) + A_t(\hat{u})$ ,  $0 \leq t < T$ , the corresponding cost process from Lemma 4.1. We will now show that  $\hat{u} \in \mathcal{U}$  and that  $\hat{u}$  attains the lower bound in (57), i.e.,

$$J(\hat{u}) = v$$

finishing our verification argument. Indeed, first note that, by choice of  $\hat{u}$ , we have

$$A_t(\hat{u}) = \int_0^t (\xi_s - \hat{\xi}_s)^2 \nu_s ds + \int_0^t c_s d[\hat{\xi}]_s \quad (0 \leq t < T)$$

and, in particular, the relation

$$v = c_0(x - \hat{\xi}_0)^2 + \mathbb{E}[A_{T-}(\hat{u})] < \infty.$$

Next, since  $C(\hat{u})$  is a non-negative local submartingale on  $[0, T]$  by virtue of Lemma 4.1 above, let us again consider a localizing sequence of stopping times  $(\hat{\tau}^n)_{n=1,2,\dots}$  such that  $\hat{\tau}^n \uparrow T$   $\mathbb{P}$ -a.s. strictly from below for  $n \rightarrow \infty$  and such that  $(M_{t \wedge \hat{\tau}^n}(\hat{u}))_{0 \leq t < T}$  is a uniformly integrable martingale for each  $n$ . Then, for any stopping time  $\tau < T$   $\mathbb{P}$ -a.s., application of Fatou's Lemma and once more Doob's Optional Sampling Theorem yields

$$\begin{aligned} \mathbb{E}[C_\tau(\hat{u})] &= \mathbb{E}[\liminf_{n \rightarrow \infty} C_{\tau \wedge \hat{\tau}^n}(\hat{u})] \leq \liminf_{n \rightarrow \infty} \mathbb{E}[C_{\tau \wedge \hat{\tau}^n}(\hat{u})] \\ &= c_0(x - \hat{\xi}_0)^2 + \liminf_{n \rightarrow \infty} \{ \mathbb{E}[A_{\tau \wedge \hat{\tau}^n}(\hat{u})] + \mathbb{E}[M_{\tau \wedge \hat{\tau}^n}(\hat{u})] \} \\ &= c_0(x - \hat{\xi}_0)^2 + \liminf_{n \rightarrow \infty} \mathbb{E}[A_{\tau \wedge \hat{\tau}^n}(\hat{u})] \\ &= c_0(x - \hat{\xi}_0)^2 + \mathbb{E}[A_\tau(\hat{u})] \leq c_0(x - \hat{\xi}_0)^2 + \mathbb{E}[A_{T-}(\hat{u})] = v, \end{aligned}$$

where we also used monotone convergence as well as the fact that  $(A(\hat{u})_t)_{0 \leq t < T}$  is an increasing process  $\mathbb{P}$ -a.s. Hence, it holds that

$$J(\hat{u}) = \limsup_{\tau \uparrow T} \mathbb{E}[C_\tau(\hat{u})] \leq v < \infty \quad (58)$$

and thus  $\hat{u} \in \mathcal{U}$ . In particular, due to (57), we actually have  $J(\hat{u}) = v$  as desired. It is left to argue that  $\hat{X}_T = x + \int_0^T \hat{u}_t dt$  exists  $\mathbb{P}$ -a.s. Indeed,  $\hat{u} \in \mathcal{U}$  implies that  $\mathbb{E}[\int_0^T \hat{u}_t^2 \kappa_t dt] < \infty$ . Application of Cauchy-Schwarz inequality together with condition (6) yields that  $\hat{u} \in L^1([0, T], dt)$   $\mathbb{P}$ -a.s.

Finally, let us assume that (30) is satisfied. Then, it follows from (56) and (58) that  $\hat{u} \in \mathcal{U}$ , i.e.,  $\mathcal{U} \neq \emptyset$ . In other words, condition (30) is sufficient.  $\square$

The final Lemma justifies the interpretation in Remark 2.6:

**Lemma 4.2.** *Let us assume that  $\lim_{t \uparrow T} L_t = L_T$  in  $L^1(\mathbb{P})$  and that  $\nu \in L^1(\mathbb{P} \otimes dt)$ . Then, we have  $d\mathbb{P} \otimes dt$ -a.e. on  $\Omega \times [0, T)$  the representation*

$$c_t = \mathbb{E} \left[ L_T e^{\int_0^t \frac{c_u}{\kappa_u} du} + \int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right]. \quad (59)$$

Moreover, we have the identity

$$\mathbb{E} \left[ \int_t^T \frac{e^{-\int_t^r \frac{c_u}{\kappa_u} du}}{(1 - w_t)c_t} \nu_r dr \middle| \mathcal{F}_t \right] = 1 \quad d\mathbb{P} \otimes dt\text{-a.e. on } \Omega \times [0, T), \quad (60)$$

where the weight process  $(w_t)_{0 \leq t < T}$  defined in (22) satisfies  $0 \leq w_t < 1$   $\mathbb{P}$ -a.s. for all  $0 \leq t < T$ .

*Proof.* Recall the dynamics of the process  $(L_t)_{0 \leq t < T}$  in (16), i.e.,

$$L_t = c_0 - \int_0^t e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr - \int_0^t e^{-\int_0^r \frac{c_u}{\kappa_u} du} dN_r \quad (0 \leq t < T).$$

Hence, for all  $0 \leq t \leq s < T$  we may write

$$L_s - L_t = - \int_t^s e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr - \int_t^s e^{-\int_0^r \frac{c_u}{\kappa_u} du} dN_r. \quad (61)$$

Observe that the stochastic integral in (61) is a martingale on  $[0, T)$  by Assumption 2.2, property (11). Thus, applying conditional expectation to the identity in (61) yields  $d\mathbb{P} \otimes dt$ -a.e. on  $\Omega \times [0, T)$  that

$$\mathbb{E}[L_s | \mathcal{F}_t] - L_t = - \mathbb{E} \left[ \int_t^s e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right]. \quad (62)$$

Passing to the limit  $s \uparrow T$  in (62) we obtain, due to monotone convergence and due to the assumption that  $L_s$  converges in  $L^1(\mathbb{P})$  to  $L_T$ , the representation

$$L_t = \mathbb{E} \left[ L_T + \int_t^T e^{-\int_0^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right] \quad d\mathbb{P} \otimes dt\text{-a.e. on } \Omega \times [0, T). \quad (63)$$

In other words, using that  $L_t = c_t e^{-\int_0^t \frac{c_u}{\kappa_u} du}$ , we can write

$$c_t = \mathbb{E} \left[ L_T e^{\int_0^t \frac{c_u}{\kappa_u} du} + \int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right] \quad d\mathbb{P} \otimes dt\text{-a.e. on } \Omega \times [0, T)$$

as desired for (59). Finally, by definition of the weight process  $(w_t)_{0 \leq t < T}$  in (22) together with the identity in (59) we can write

$$\begin{aligned} w_t &= \frac{\mathbb{E}[L_T | \mathcal{F}_t]}{L_t} = \frac{e^{\int_0^t \frac{c_u}{\kappa_u} du}}{c_t} \mathbb{E}[L_T | \mathcal{F}_t] = \frac{1}{c_t} \mathbb{E} \left[ e^{\int_0^t \frac{c_u}{\kappa_u} du} L_T \middle| \mathcal{F}_t \right] \\ &= \frac{1}{c_t} \left( c_t - \mathbb{E} \left[ \int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right] \right) \\ &= 1 - \frac{1}{c_t} \mathbb{E} \left[ \int_t^T e^{-\int_t^r \frac{c_u}{\kappa_u} du} \nu_r dr \middle| \mathcal{F}_t \right] \quad \text{for all } 0 \leq t < T, \end{aligned} \quad (64)$$

which yields our claim (60). In particular, representation (64) also reveals that  $0 \leq w_t < 1$   $\mathbb{P}$ -a.s. for all  $0 \leq t < T$ . Finally, note that the representation in (59) also holds true in the case  $\eta = 0$   $\mathbb{P}$ -a.s. which implies  $L_T = 0$   $\mathbb{P}$ -a.s. by Lemma 2.4.  $\square$

## Appendix

Let us briefly discuss the integrability condition (13) in our standing Assumption 2.2. This condition is not regularly discussed in the BSRDE literature and thus calls for a verification in some sufficiently generic setting. So let us place ourselves in the context of Kruse and Popier [15]. We assume that the underlying filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is quasi-left continuous and we let  $(\nu_t)_{0 \leq t \leq T}$  be a nonnegative and  $(\kappa_t)_{0 \leq t \leq T}$  a strictly positive progressively measurable process which satisfy

$$\mathbb{E} \left[ \int_0^T (\kappa_t^2 + \nu_t^2) dt \right] < \infty \quad \text{and} \quad \mathbb{E} \left[ \int_0^T \frac{1}{\kappa_t} dt \right] < \infty. \quad (65)$$

Under these conditions, it follows from Corollary 1 in [15] that there exists a pair  $(c, N)$  which satisfies the BSRDE dynamics in (9) on  $[0, T)$  with the slightly weaker singular terminal condition

$$\liminf_{t \uparrow T} c_t \geq \eta \quad \mathbb{P}\text{-a.s.} \quad (66)$$

for some  $\eta \geq 0$   $\mathbb{P}$ -a.s. with  $\mathbb{P}[\eta = +\infty] > 0$  instead of (10) (but this is not crucial for our present discussion; for (10) to hold true, further assumptions on  $\eta$  and  $(\kappa_t)_{0 \leq t \leq T}$  are needed, cf. Popier [20]). As shown in [15], the solution pair  $(c, N)$  satisfies (11), that is,

$$\mathbb{E} \left[ \sup_{s \in [0, t]} |c_s|^2 + [N]_t \right] < \infty \text{ for all } 0 \leq t < T \quad (67)$$

and for any  $t \in [0, T)$  we have the estimates

$$0 \leq c_t \leq \frac{1}{(T-t)^2} \mathbb{E} \left[ \int_t^T (\kappa_s + (T-s)^2 \nu_s) ds \middle| \mathcal{F}_t \right] \quad \mathbb{P}\text{-a.s.}; \quad (68)$$

cf. Proposition 3 and Remark 4 as well as Corollary 1 in [15] with  $p = 2$ . In addition to that, we can derive the following lower bound:

**Lemma 4.3.** *For all  $t \in [0, T)$  we have*

$$c_t \geq \mathbb{E} \left[ \frac{1}{\int_t^T \frac{1}{\kappa_s} ds + \frac{1}{\eta}} \middle| \mathcal{F}_t \right] \quad \mathbb{P}\text{-a.s.} \quad (69)$$

*In particular, we have  $c_t > 0$   $\mathbb{P}$ -a.s. for all  $0 \leq t < T$ .*

*Proof.* We will adopt the same idea as in the proof of Lemma 11 in Popier [19] in the case  $\kappa \equiv 1$  (and  $\nu \equiv 0$ ). For all  $n \geq 1$  we define the processes

$$\Gamma_t^n \triangleq \mathbb{E} \left[ \frac{1}{\int_t^T \frac{1}{\kappa_s} ds + \frac{1}{\eta \wedge n}} \middle| \mathcal{F}_t \right] \quad (0 \leq t \leq T).$$

Note that  $\Gamma^n$  is well defined because the term in the conditional expectation is bounded by  $n$ . Moreover, we have pathwise the identity

$$\frac{1}{\int_t^T \frac{1}{\kappa_s} ds + \frac{1}{\eta \wedge n}} = \eta \wedge n - \int_t^T \frac{1}{\kappa_s} \left( \frac{1}{\int_s^T \frac{1}{\kappa_u} du + \frac{1}{\eta \wedge n}} \right)^2 ds \quad (0 \leq t \leq T).$$

Thus, the process  $\Gamma^n$  verifies

$$\begin{aligned}\Gamma_t^n &= \mathbb{E} \left[ \eta \wedge n - \int_t^T \frac{1}{\kappa_s} \left( \frac{1}{\int_s^T \frac{1}{\kappa_u} du + \frac{1}{\eta \wedge n}} \right)^2 ds \middle| \mathcal{F}_t \right] \\ &= \mathbb{E} \left[ \eta \wedge n - \int_t^T \frac{1}{\kappa_s} ((\Gamma_s^n)^2 + U_s^n) ds \middle| \mathcal{F}_t \right] \quad (0 \leq t \leq T)\end{aligned}$$

with adapted process  $U^n$  given by

$$U_s^n \triangleq \mathbb{E} \left[ \frac{1}{\left( \int_s^T \frac{1}{\kappa_u} du + \frac{1}{\eta \wedge n} \right)^2} \middle| \mathcal{F}_s \right] - (\Gamma_s^n)^2 \quad (0 \leq s \leq T).$$

Now, observe that  $U_s^n \geq 0$  for all  $0 \leq s \leq T$  due to Jensen's inequality. Thus, the comparison result in Kruse and Popier [16], Proposition 4, together with the construction of the solution process  $(c_t)_{0 \leq t < T}$  via a truncation procedure in [15], finally yields that for all  $0 \leq t < T$  we have

$$c_t \geq \mathbb{E} \left[ \frac{1}{\int_t^T \frac{1}{\kappa_s} ds + \frac{1}{\eta \wedge n}} \middle| \mathcal{F}_t \right] \quad \mathbb{P}\text{-a.s.}$$

Appealing to the monotone convergence theorem for  $n \rightarrow \infty$  yields the desired result.  $\square$

From now on, still within the context of [15], let us for simplicity further confine ourselves to a Brownian framework (that is, the underlying filtration  $(\mathcal{F}_t)_{0 \leq t \leq T}$  is the completed filtration generated by a Brownian motion) and let us make besides (65) the following additional assumptions on  $(\nu_t)_{0 \leq t \leq T}$ ,  $(\kappa_t)_{0 \leq t \leq T}$  and  $\eta$ : We assume that the process  $(\kappa_t)_{0 \leq t \leq T}$  is bounded from below and above, i.e., it holds that

$$0 < k \leq \kappa_t \leq K < \infty \quad (0 \leq t \leq T) \quad (70)$$

for some constants  $k, K \in \mathbb{R}$ . Moreover, we assume that the process

$$\frac{1}{T-t} \mathbb{E} \left[ \int_t^T (T-s)^2 \nu_s ds \middle| \mathcal{F}_t \right] \leq C \quad (0 \leq t \leq T) \quad (71)$$

for some constant  $C < \infty$ . Finally, we assume that there exists a constant  $\varepsilon > 0$  such that

$$\mathbb{P}[\varepsilon \leq \eta \leq +\infty] = 1. \quad (72)$$

Then, the following holds true.

**Lemma 4.4.** *Under the conditions (65), (70), (71) and (72) the solution process  $(c_t)_{0 \leq t < T}$  to the BSRDE in (9) on  $[0, T)$  with singular terminal condition (66) satisfies*

$$\int_0^T \frac{d\langle c \rangle_t}{c_t^2} < \infty \quad \text{on the set } \{\eta = +\infty\},$$

i.e., condition (13) holds true.

*Proof.* We extend the proof of Proposition 10 in Popier [19] done for the specific case  $\kappa \equiv 1$  and  $\nu \equiv 0$  to our more general setting by using the upper and lower bounds of the process  $(c_t)_{0 \leq t < T}$  in (68) and (69). First, note that assumptions (70) and (72) imply for the lower bound in (69) that

$$c_t \geq \frac{k\varepsilon}{(T-t)\varepsilon + k} \quad (0 \leq t < T). \quad (73)$$

Concerning the upper bound in (68), we obtain due to (70) and (71)

$$c_t \leq \frac{K + \text{const}}{T-t} \quad (0 \leq t < T). \quad (74)$$

Since the process  $c$  is bounded from below on  $[0, T]$ , we can apply Itô's formula on  $[0, T - \delta]$  for some  $0 < \delta < T$  to the process  $\sqrt{(T-t)c_t}$ . Using the BSRDE dynamics of  $c$  in (9), we obtain

$$\begin{aligned} 0 &\leq \sqrt{(T-t)c_t} \\ &= \sqrt{Tc_0} + \int_0^t \left( \frac{\sqrt{T-s}}{2\sqrt{c_s}} \left( \frac{c_s^2}{\kappa_s} - \nu_s \right) - \frac{\sqrt{c_s}}{2\sqrt{T-s}} \right) ds \\ &\quad - \frac{1}{8} \int_0^t \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s - \frac{1}{2} \int_0^t \frac{\sqrt{T-s}}{\sqrt{c_s}} dN_s \\ &= \sqrt{Tc_0} + \frac{1}{2} \int_0^t \sqrt{T-s} \frac{\sqrt{c_s}}{\kappa_s} \left( c_s - \frac{\nu_s \kappa_s}{c_s} - \frac{\kappa_s}{T-s} \right) ds \\ &\quad - \frac{1}{8} \int_0^t \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s - \frac{1}{2} \int_0^t \frac{\sqrt{T-s}}{\sqrt{c_s}} dN_s \quad (0 \leq t \leq T - \delta) \end{aligned}$$

and hence

$$\begin{aligned} & \frac{1}{8} \int_0^{T-\delta} \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s + \frac{1}{2} \int_0^{T-\delta} \frac{\sqrt{T-s}}{\sqrt{c_s}} dN_s \\ & \leq \sqrt{Tc_0} + \frac{1}{2} \int_0^{T-\delta} \sqrt{T-s} \frac{\sqrt{c_s}}{\kappa_s} \left( c_s - \frac{\nu_s \kappa_s}{c_s} - \frac{\kappa_s}{T-s} \right) ds \end{aligned} \quad (75)$$

for all  $0 < \delta < T$ . Observe that due to the boundedness of  $c$  in (73) and (74) and  $\kappa$  in (70) as well as the integrability assumption on  $\nu$  in (65) it holds for all  $0 < \delta < T$  that

$$\begin{aligned} & \mathbb{E} \left[ \int_0^{T-\delta} \sqrt{T-s} \frac{\sqrt{c_s}}{\kappa_s} \left| c_s - \frac{\nu_s \kappa_s}{c_s} - \frac{\kappa_s}{T-s} \right| ds \right] \\ & \leq \text{const} \mathbb{E} \left[ \int_0^{T-\delta} \left| c_s - \frac{\nu_s \kappa_s}{c_s} - \frac{\kappa_s}{T-s} \right| ds \right] \\ & \leq \text{const} \left( \mathbb{E} \left[ \int_0^{T-\delta} c_s ds \right] + \mathbb{E} \left[ \int_0^{T-\delta} \frac{\nu_s \kappa_s}{c_s} ds \right] + \mathbb{E} \left[ \int_0^{T-\delta} \frac{\kappa_s}{T-s} ds \right] \right) < \infty. \end{aligned}$$

Hence, by using the upper bound on  $c$  in (68) and Fubini's Theorem, we can compute

$$\begin{aligned} & \mathbb{E} \left[ \int_0^{T-\delta} \left( c_s - \frac{\nu_s \kappa_s}{c_s} - \frac{\kappa_s}{T-s} \right) ds \right] \leq \mathbb{E} \left[ \int_0^{T-\delta} \left( c_s - \frac{\kappa_s}{T-s} \right) ds \right] \\ & \leq \mathbb{E} \left[ \int_0^{T-\delta} \left( \frac{1}{(T-s)^2} \mathbb{E} \left[ \int_s^T (\kappa_u + (T-u)^2 \nu_u) du \middle| \mathcal{F}_s \right] - \frac{\kappa_s}{T-s} \right) ds \right] \\ & \leq \mathbb{E} \left[ \int_0^{T-\delta} \frac{1}{(T-s)^2} \left( \int_s^T \kappa_u du \right) ds - \int_0^{T-\delta} \frac{\kappa_s}{T-s} ds \right] \\ & \quad + \mathbb{E} \left[ \int_0^{T-\delta} \frac{1}{(T-s)^2} \left( \int_s^T (T-u)^2 \nu_u du \right) ds \right]. \end{aligned} \quad (76)$$

Using once more Fubini's Theorem and the fact that  $\kappa_t \leq K$  for all  $0 \leq t \leq T$ , we get for the first expectation in (76) the estimate

$$\begin{aligned} & \mathbb{E} \left[ \int_0^{T-\delta} \frac{1}{(T-s)^2} \left( \int_s^T \kappa_u du \right) ds - \int_0^{T-\delta} \frac{\kappa_s}{T-s} ds \right] \\ & = \mathbb{E} \left[ \int_0^{T-\delta} \frac{\kappa_u}{T-u} du + \int_{T-\delta}^T \frac{\kappa_u}{\delta} du - \frac{1}{T} \int_0^T \kappa_u du - \int_0^{T-\delta} \frac{\kappa_s}{T-s} ds \right] \\ & \leq K. \end{aligned} \quad (77)$$

Concerning the second expectation in (76), application of Fubini's Theorem yields

$$\begin{aligned} & \mathbb{E} \left[ \int_0^{T-\delta} \frac{1}{(T-s)^2} \left( \int_s^T (T-u)^2 \nu_u du \right) ds \right] \\ & \leq \mathbb{E} \left[ \int_0^{T-\delta} (T-u) \nu_u du + \delta \int_{T-\delta}^T \nu_u du \right]. \end{aligned} \quad (78)$$

Consequently, taking expectation in (75) and using that the stochastic integral with respect to  $N$  in (75) is a true martingale on  $[0, T-\delta]$  due to (73) and (67), we obtain together with the estimates in (77) and (78) the upper bound

$$\begin{aligned} & \frac{1}{8} \mathbb{E} \left[ \int_0^{T-\delta} \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s \right] \\ & \leq \sqrt{Tc_0} + \text{const} \left( K + \mathbb{E} \left[ \int_0^{T-\delta} (T-u) \nu_u du + \delta \int_{T-\delta}^T \nu_u du \right] \right). \end{aligned}$$

Passing to the limit  $\delta \downarrow 0$  we get with monotone convergence

$$\begin{aligned} & \mathbb{E} \left[ \int_0^T \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s \right] \\ & \leq 8 \left( \sqrt{Tc_0} + \text{const} \left( K + \mathbb{E} \left[ \int_0^T (T-u) \nu_u du \right] \right) \right) < \infty, \end{aligned} \quad (79)$$

due to (65). Now, using (69), observe that we can further estimate the process  $(c_t)_{0 \leq t < T}$  from below by

$$\begin{aligned} c_s & \geq \mathbb{E} \left[ \frac{1}{\int_s^T \frac{1}{\kappa_u} du + \frac{1}{\eta}} \middle| \mathcal{F}_s \right] \geq \mathbb{E} \left[ \frac{1}{\int_s^T \frac{1}{\kappa_u} du + \frac{1}{\eta}} 1_{\{\eta = +\infty\}} \middle| \mathcal{F}_s \right] \\ & = \mathbb{E} \left[ \frac{1}{\int_s^T \frac{1}{\kappa_u} du} 1_{\{\eta = +\infty\}} \middle| \mathcal{F}_s \right] \geq \frac{k}{T-s} \mathbb{E} [1_{\{\eta = +\infty\}} | \mathcal{F}_s]. \end{aligned}$$

Plugging back this lower bound into the left hand side of (79) and using



optional projection, we get

$$\begin{aligned}
\infty &> \mathbb{E} \left[ \int_0^T \frac{\sqrt{T-s}}{c_s^{3/2}} d\langle c \rangle_s \right] = \mathbb{E} \left[ \int_0^T \frac{\sqrt{T-s}}{c_s^2} \sqrt{c_s} d\langle c \rangle_s \right] \\
&\geq \sqrt{k} \mathbb{E} \left[ \int_0^T \frac{1}{c_s^2} \mathbb{E} [1_{\{\eta=+\infty\}} | \mathcal{F}_s] d\langle c \rangle_s \right] = \sqrt{k} \mathbb{E} \left[ \int_0^T \frac{1}{c_s^2} 1_{\{\eta=+\infty\}} d\langle c \rangle_s \right] \\
&= \sqrt{k} \mathbb{E} \left[ 1_{\{\eta=+\infty\}} \left( \int_0^T \frac{1}{c_s^2} d\langle c \rangle_s \right) \right],
\end{aligned}$$

which yields the desired result.  $\square$

## References

- [1] Robert Almgren and Tianhui Michael Li. Option hedging with smooth market impact. Preprint, April 2016.
- [2] Stefan Ankirchner and Thomas Kruse. Optimal position targeting with stochastic linear-quadratic costs. In *Advances in Mathematics of Finance*, volume 104 of *Banach Center Publ.*, pages 9–24. Polish Acad. Sci. Inst. Math., Warsaw, 2015.
- [3] Stefan Ankirchner, Monique Jeanblanc, and Thomas Kruse. BSDEs with singular terminal condition and a control problem with constraints. *SIAM Journal on Control and Optimization*, 52(2):893–913, 2014. doi: 10.1137/130923518. URL <http://dx.doi.org/10.1137/130923518>.
- [4] Peter Bank, H. Mete Soner, and Moritz Voß. Hedging with temporary price impact. *Mathematics and Financial Economics*, pages 1–25, 2016. doi: 10.1007/s11579-016-0178-4. URL <http://dx.doi.org/10.1007/s11579-016-0178-4>.
- [5] Jean-Michel Bismut. Linear quadratic optimal stochastic control with random coefficients. *SIAM Journal on Control and Optimization*, 14(3):419–444, 1976. doi: 10.1137/0314028. URL <http://dx.doi.org/10.1137/0314028>.
- [6] Jean-Michel Bismut. Controle des systemes lineaires quadratiques : Applications de l’integrale stochastique. In C. Dellacherie, P. A. Meyer, and

- M. Weil, editors, *Séminaire de Probabilités XII: Université de Strasbourg 1976/77*, pages 180–264. Springer Berlin Heidelberg, Berlin, Heidelberg, 1978. ISBN 978-3-540-35856-5. doi: 10.1007/BFb0064606. URL <http://dx.doi.org/10.1007/BFb0064606>.
- [7] Jiatu Cai, Mathieu Rosenbaum, and Peter Tankov. Asymptotic lower bounds for optimal tracking: a linear programming approach. Preprint, October 2015.
  - [8] Álvaro Cartea and Sebastian Jaimungal. A closed-form execution strategy to target VWAP. Preprint, January 2015.
  - [9] Patrick Chan and Ronnie Sircar. Optimal trading with predictable return and stochastic volatility. Preprint, July 2016.
  - [10] Christoph Frei and Nicholas Westray. Optimal execution of a VWAP order: A stochastic control approach. *Mathematical Finance*, 2013. doi: 10.1111/mafi.12048.
  - [11] Paulwin Graewe, Ulrich Horst, and Jinniao Qiu. A non-markovian liquidation problem and backward SPDEs with singular terminal conditions. *SIAM Journal on Control and Optimization*, 53(2):690–711, 2015. doi: 10.1137/130944084. URL <http://dx.doi.org/10.1137/130944084>.
  - [12] Shaolin Ji and Xun Yu Zhou. A maximum principle for stochastic optimal control with terminal state constraints, and its applications. *Commun. Inf. Syst.*, 6(4):321–338, 2006. URL <http://projecteuclid.org/euclid.cis/1183729000>.
  - [13] Ioannis Karatzas and Steven E. Shreve. *Brownian motion and stochastic calculus*, volume 113 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, second edition, 1991. ISBN 0-387-97655-8.
  - [14] Michael Kohlmann and Shanjian Tang. Global adapted solution of one-dimensional backward stochastic Riccati equations, with application to the mean-variance hedging. *Stochastic Processes and their Applications*, 97(2):255 – 288, 2002. ISSN 0304-4149. doi: [http://dx.doi.org/10.1016/S0304-4149\(01\)00133-8](http://dx.doi.org/10.1016/S0304-4149(01)00133-8).

- [15] Thomas Kruse and Alexandre Popier. Minimal supersolutions for BSDEs with singular terminal condition and application to optimal position targeting. *Stochastic Processes and their Applications*, 126(9):2554 – 2592, 2016. ISSN 0304-4149. doi: <http://dx.doi.org/10.1016/j.spa.2016.02.010>. URL <http://www.sciencedirect.com/science/article/pii/S0304414916000417>.
- [16] Thomas Kruse and Alexandre Popier. BSDEs with monotone generator driven by Brownian and Poisson noises in a general filtration. *Stochastics*, 88(4):491–539, 2016. doi: 10.1080/17442508.2015.1090990. URL <http://dx.doi.org/10.1080/17442508.2015.1090990>.
- [17] Felix Naujokat and Nicholas Westray. Curve following in illiquid markets. *Mathematics and Financial Economics*, 4(4):299–335, 2011. ISSN 1862-9679. doi: 10.1007/s11579-011-0042-5. URL <http://dx.doi.org/10.1007/s11579-011-0042-5>.
- [18] Huy  n Pham. *Continuous-time Stochastic Control and Optimization with Financial Applications*. Stochastic Modelling and Applied Probability. Springer, 2009. ISBN 9783540895008. URL <http://books.google.de/books?id=xBsagiBp1SYC>.
- [19] Alexandre Popier. Backward stochastic differential equations with singular terminal condition. *Stochastic Processes and their Applications*, 116(12):2014 – 2056, 2006. ISSN 0304-4149. doi: <http://dx.doi.org/10.1016/j.spa.2006.05.012>. URL <http://www.sciencedirect.com/science/article/pii/S0304414906000822>.
- [20] Alexandre Popier. Limit behaviour of BSDE with jumps and with singular terminal condition. Preprint, 2016.
- [21] Philip E. Protter. *Stochastic integration and differential equations*, volume 21 of *Applications of Mathematics (New York)*. Springer-Verlag, Berlin, second edition, 2004. ISBN 3-540-00313-4. Stochastic Modelling and Applied Probability.
- [22] L. C. G. Rogers and Surbjeet Singh. The cost of illiquidity and its effects on hedging. *Mathematical Finance*, 20:597–615, 2010.
- [23] Alexander Schied. A control problem with fuel constraint and Dawson-Watanabe superprocesses. *Ann. Appl. Probab.*, 23

(6):2472–2499, 12 2013. doi: 10.1214/12-AAP908. URL  
<http://dx.doi.org/10.1214/12-AAP908>.

- [24] Jiongmin Yong and Xun Yu Zhou. *Stochastic controls: Hamiltonian systems and HJB equations*. Springer, Berlin, 1999.